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The effect of initial conditions and lateral boundaries on convection

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A theoretical analysis of two-dimensional, finite-amplitude, thermal convection is made for a fluid which has an infinite Prandtl number. The vertical velocity disturbance is expanded in a double Fourier series which satisfies the horizontal and lateral boundary conditions. The resulting coupled sets of non-linear differential equations are solved numerically. It is found that for a particular Rayleigh number the number and size of the convection cells that form depend upon the ratio of the distance between the lateral boundaries to the depth of the fluid layer and on the initial conditions. The steady-state solutions are not unique and the solution for which the heat transport is a maximum is not necessarily the solution that results. Where there are no lateral boundaries, the lateral edges of the cells tend to tilt and the Nusselt number increases slightly.

1. Introduction

In the theoretical investigation of convection the effects of initial conditions and lateral boundaries are usually neglected. One usually assumes that the steady-state solutions, at least for small Rayleigh numbers, are unique, and therefore one is justified in starting with any initial conditions. In most investigations of convection it is also assumed that the fluid is infinite in horizontal extent. Then there is no horizontal length scale imposed by lateral boundaries and the wavelength of periodic cellular convection is an unknown parameter, which must be determined. So far no one has been able to derive this wavelength from the hydrodynamic and thermodynamic equations except in the trivial case at the critical Rayleigh number where only one wavelength is unstable.

The most widely used criterion that has been used to select a wavelength for convection is the hypothesis proposed by Malkus (1954), that the convecting fluid should transport a maximum amount of heat. Recent studies of convection by Herring (1963, 1964) and Veronis (1966) have made use of this hypothesis to select wavelengths for plots of Nusselt number *versus* Rayleigh number. It seems feasible that under the turbulent conditions at large Rayleigh numbers envisaged by Malkus the hypothesis may find proof in statistical mechanical considerations. This has not yet been accomplished. At small Rayleigh numbers convection sets in as an organized secondary flow, and the validity of a thermodynamic principle in determining the periodicity of the flow appears questionable. The validity of the equivalent principle of maximum dissipation in cylindrical Couette flow has recently been challenged by Meyer (1967); however, in order to select a periodicity length he has to make an additional assumption, that the preferred length is that one which results in uniform cells.

In Herring's calculations all non-linear interactions are neglected except those which represent interactions of the fluctuations with the horizontally averaged temperature field. This is equivalent to expanding the horizontal dependence of the vertical velocity in a Fourier series in which only the sine terms are used. In Veronis's calculations the complete set of non-linear interactions are retained. He uses a Fourier cosine expansion for the horizontal dependence of the vertical velocity, which is appropriate for a region confined by free-insulating lateral boundaries. However, since he is only interested in single-cell solutions in the steady state, he uses a diagonalized two-dimensional Fourier expansion in which all the odd coefficients are omitted. Thus his method is not applicable to the investigation of the effects of initial conditions, nor can it be used to investigate the transitions to multiple-cell solutions.

The aim of the present paper is to investigate the factors which govern the number and size of convection cells that will form under specified conditions for two-dimensional flow. Thus we cannot neglect the fluctuating self-interactions, as Herring did, since these interactions are probably important in the wavelength selection mechanism, nor can we use the very efficient expansion scheme used by Veronis since it is only applicable to single-cell solutions. The complete expansions, however, result in extremely complicated sets of equations for the Fourier coefficients, which can be solved economically only for a very restricted class of conditions even with a high-speed electronic computer. Accordingly, it was decided to solve the equations only for the limiting case of very large Prandtl number. Then the fluctuating self-interactions only appear in the heat-conduction equation and the resulting sets of equations for the Fourier coefficients are considerably simpler than those used by Veronis. It is believed that this simplification gives a good description for fluids which have large Prandtl numbers and provides at least a qualitatively correct description of fluids with Prandtl numbers greater than unity (Kraichnan (1962) has discussed the effect of Prandtl number on convection at some length). We will also give particular attention to initial conditions and concentrate mainly on initially infinitesimal 'white noise' disturbances, which should not prejudice our solutions, and which therefore should be similar to the initial disturbances that occur in laboratory experiments.

2. The basic equations

We will restrict our investigation to an incompressible fluid in which the temperature is uniform on the horizontal boundary surfaces. All motions are assumed to be two-dimensional in the (x, z)-plane. All properties of the fluid are assumed to be constant except for density as it affects the buoyancy term (the Boussinesq approximation). The equation of state is assumed to be linear. Under these conditions the Navier–Stokes equations become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)u, \tag{1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)w-\alpha gT, \tag{2}$$

and the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0, \tag{3}$$

where u is the velocity in the x-direction (horizontal); w the velocity in the z-direction (vertically downward); P the pressure; T the temperature with respect to some reference temperature; ρ the density at the reference temperature; α the coefficient of thermal volume expansion; ν the kinematic viscosity; and g the acceleration of gravity. We will also need the source-free heat conduction equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)T, \tag{4}$$

where κ is the thermometric conductivity.

If we differentiate (1) by z and (2) by x and subtract, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\left(u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) = \nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\alpha g\frac{\partial T}{\partial x}. \tag{5}$$

It is convenient to write our equations in dimensionless form using for the length scale the depth of the fluid layer h, for the time scale h^2/κ, and for the temperature scale, the difference in temperature between the top and bottom surfaces ΔT. It is also convenient to divide the temperature T into a horizontal mean, $\overline{T}(z,t)$, and a fluctuating part, $\theta(x,z,t)$, so that $T = \overline{T}+\theta$ and $\overline{\theta} = 0$. Equations (4) and (5) then become

$$\frac{\partial \overline{T}}{\partial t}+\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial z}+w\frac{\partial \overline{T}}{\partial z} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\theta+\frac{\partial^2 \overline{T}}{\partial z^2}, \tag{6}$$

and

$$\frac{1}{p}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{1}{p}\left(u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+R\frac{\partial \theta}{\partial x}, \tag{7}$$

where

$$p = \nu/\kappa, \tag{8}$$

is the Prandtl number and

$$R = \alpha g\Delta T h^3/\kappa\nu, \tag{9}$$

is the Rayleigh number.

We will now assume that we are dealing with a fluid which has a Prandtl number so large that it can be considered infinite. Thus (7) can be written

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) = -R\frac{\partial \theta}{\partial x}. \tag{10}$$

It will also be convenient to use the equation obtained from (10) by differentiating with respect to x,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)^2 w = R\frac{\partial^2 \theta}{\partial x^2}. \tag{11}$$

3. Boundary conditions

We will assume that the horizontal boundary surfaces at $z = 0$ and $z = 1$ are perfectly flat and fixed but not capable of supporting tangential stresses, the so-called 'free boundary conditions'; thus

$$w = \partial^2 w/\partial z^2 = 0 \quad \text{at} \quad z = 0, 1. \tag{12}$$

If we further assume that the horizontal boundary surfaces are perfectly conducting so that the horizontal fluctuating part of the temperature vanishes there, then we have

$$\theta = 0 \quad \text{at} \quad z = 0, 1. \tag{13}$$

Equation (11) then gives the further condition on w that

$$\partial^4 w/\partial z^4 = 0 \quad \text{at} \quad z = 0, 1. \tag{14}$$

We will consider two different configurations with respect to lateral boundary conditions. In the first case the surfaces at $x = 0$ and $x = L$ are 'free' so that

$$u = \partial^2 u/\partial x^2 = 0 \quad \text{at} \quad x = 0, L, \tag{15}$$

and perfectly insulating so that

$$\partial \theta/\partial x = 0 \quad \text{at} \quad x = 0, L. \tag{16}$$

Equations (3) and (10) then give the further condition on w that

$$\partial w/\partial x = \partial^3 w/\partial x^3 = 0 \quad \text{at} \quad x = 0, L. \tag{17}$$

In the second case the fluid is assumed to be infinite in horizontal extent.

4. Solution method for finite horizontal extent

We will expand the vertical velocity, w, in a double Fourier series with time-dependent coefficients such that the boundary conditions (12), (14) and (17) are satisfied, and such that, when the continuity equation (3) is solved for u, the boundary conditions (15) are satisfied. Such an expansion is

$$w = \sum_m \sum_n A_{mn}(t) \cos(m\pi x/L) \sin(n\pi z). \tag{18}$$

Substituting (18) into (3) and solving for u, we have

$$u = -\sum_m \sum_n A_{mn}(t)(nL/m) \sin(m\pi x/L) \cos(n\pi z). \tag{19}$$

We will also substitute (18) into (11) and solve for θ. The differentiation of the Fourier series term by term is valid here since we can carry out an integration by parts and show that the differentiated series converges to a series in which the coefficients are the Fourier coefficients of the derivatives of w since the differentiated parts vanish when we apply the boundary conditions (12) and (17). In this manner we obtain

$$\theta = -\sum_m \sum_n A_{mn}(t) \frac{(n^2\pi^2 + m^2\pi^2/L^2)^2}{Rm^2\pi^2/L^2} \cos(m\pi x/L) \sin(n\pi z). \tag{20}$$

We will also expand the horizontally averaged temperature, $\bar{T}$, in a Fourier series with time-dependent coefficients. We will consider at this time only two different initial temperature profiles: the linear profile and the profile generated by suddenly changing the temperature at one of the two horizontal boundary surfaces by an increment, ΔT. For these temperature profiles an appropriate expansion is

$$\bar{T} = z - 1 + 2\sum_{l} C_l(t)\sin(l\pi z)/l\pi, \tag{21}$$

where $C_l(0) = 0$ in the linear case and $C_l(0) = 1$ in the sudden change case.

We may now substitute these expansions for w, u, θ, and $\bar{T}$ into the heat conduction equation (6). We then obtain sets of coupled ordinary differential equations for the Fourier coefficients $A_{mn}(t)$ and $C_l(t)$. First, we multiply (6) by $(1/L)\cos(k\pi x/L)\sin(r\pi z)$ and integrate from $x = 0$ to $x = L$ and from $z = 0$ to $z = 1$. This gives

$$\begin{aligned}
&-A'_{kr}(t)\,[(r^2\pi^2 + k^2\pi^2/L^2)^2/(4Rk^2\pi^2/L^2)]\\
&-\sum_m\sum_n\sum_i\sum_j A_{mn}(t)\,A_{ij}(t)\,(n\pi i/m)\,[(j^2\pi^2 + i^2\pi^2/L^2)^2/(Ri^2\pi^2/L^2)]\,I^{ssc}_{rjm}\,I^{ssc}_{mik}\\
&-\sum_m\sum_n\sum_i\sum_j A_{mn}(t)\,A_{ij}(t)\,j\pi\,[(j^2\pi^2 + i^2\pi^2/L^2)^2/(Ri^2\pi^2/L^2)]\,I^{ssc}_{rnj}\,I^{ccc}_{mik}\\
&+A_{kr}(t)/4 + \sum_n\sum_l A_{kn}(t)\,C_l(t)\,I^{ssc}_{rnl}\\
&\quad = A_{kr}(t)\,[(r^2\pi^2 + k^2\pi^2/L^2)^3/(4Rk^2\pi^2/L^2)],
\end{aligned} \tag{22}$$

where

$$I^{ssc}_{ijk} = \frac{1}{L}\int_0^L \sin\left(\frac{i\pi x}{L}\right)\sin\left(\frac{j\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)dx = \begin{cases} \tfrac{1}{4} & \text{if } k = |i-j|,\\ -\tfrac{1}{4} & \text{if } k = i+j,\\ 0 & \text{otherwise,}\end{cases}$$

and
$$I^{ccc}_{ijk} = \frac{1}{L}\int_0^L \cos\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)dx = \begin{cases} \tfrac{1}{4} & \text{if } k = |i-j|,\\ \tfrac{1}{4} & \text{if } k = i+j,\\ 0 & \text{otherwise.}\end{cases}$$

Secondly, we multiply (6) by $(1/L)\sin(r\pi z)$ and integrate from $x = 0$ to $x = L$ and from $z = 0$ to $z = 1$. This gives

$$\begin{aligned}
C'_r(t)/r\pi &- \sum_m\sum_n\sum_j A_{mn}(t)\,A_{mj}(t)\,(nL^2/2Rm^2\pi)\,(j^2\pi^2 + m^2\pi^2/L^2)^2\,I^{ssc}_{rjm}\\
&- \sum_m\sum_n\sum_j A_{mn}(t)\,A_{mj}(t)\,(jL^2/2Rm^2\pi)\,(j^2\pi^2 + m^2\pi^2/L^2)^2\,I^{ssc}_{rnj}\\
&= -C_r(t)\,r\pi.
\end{aligned} \tag{23}$$

If we now truncate the series expansions to M terms horizontally and N terms vertically, we will have $M \times N$ equations in (22) and N equations in (23). These can be solved numerically for appropriate initial conditions using the Runge–Kutta–Gill fourth-order method (Romanelli 1960, pp. 110–120).

5. Results for the linear temperature profile

Most of the calculations made in this study involved starting with a linear temperature profile, which was generated by setting the $C_l(0) = 0$ in (21). A small velocity disturbance was introduced in the fluid at $t = 0$, and the time development of the system was then followed until a steady state was obtained. The Nusselt number

$$N = Hh/k\Delta T, \tag{24}$$

where H is the heat transferred per unit area per unit time, and k is the thermal conductivity of the fluid, was calculated as a function of the Rayleigh number R, and the length ratio of the distance between the lateral boundaries to the depth

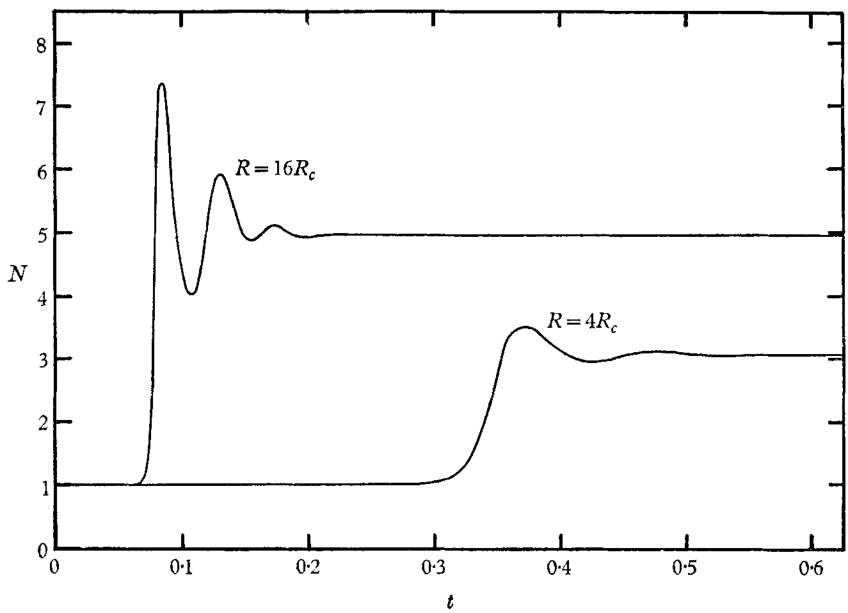

FIGURE 1. Time development of Nusselt number starting from a linear temperature profile and an infinitesimal disturbance.

of the fluid layer L. In general, the Nusselt numbers for the top and bottom surfaces were nearly equal at all times in the case of an initially linear temperature profile and exactly equal when a steady state was achieved irrespective of the initial conditions. Figure 1 shows the time development of the Nusselt number for Rayleigh numbers of 2630, which happens to be four times the critical Rayleigh number for free-conducting boundaries for a layer of infinite horizontal extent ($4R_c$), and 10,520 ($16R_c$) at a length ratio L of 1·3. In each case the initial velocity disturbance at $t = 0$ was essentially 'white noise'; that is, all the Fourier components of the velocity disturbance were equal and very small compared to the final amplitude of the largest component. In the present case initial amplitudes on the order of 10^{-6} were used and thus were effectively infinitesimal compared to the largest final amplitudes, which were on the order of 10^1. More will be said on this point later in this section.

The number of terms in the Fourier expansion that is necessary to give accurate results is a function of Rayleigh number and length ratio. In general, a sufficient number of terms was used so that the Nusselt number was within 1% of that of the next higher approximation. To meet this requirement it was found that the number of horizontal terms necessary for any particular Rayleigh number was roughly proportional to the product of the length ratio with the number of vertical terms. An IBM 7094-7040 Direct Coupled System was used for all computations. The memory of the computer limited accurate calculations to Rayleigh numbers less than about 20,000. In a typical calculation for $R = 4R_c$, $N = 6$ and $M = 6$ it took approximately 10 minutes of computation time to reach a reasonably steady state.

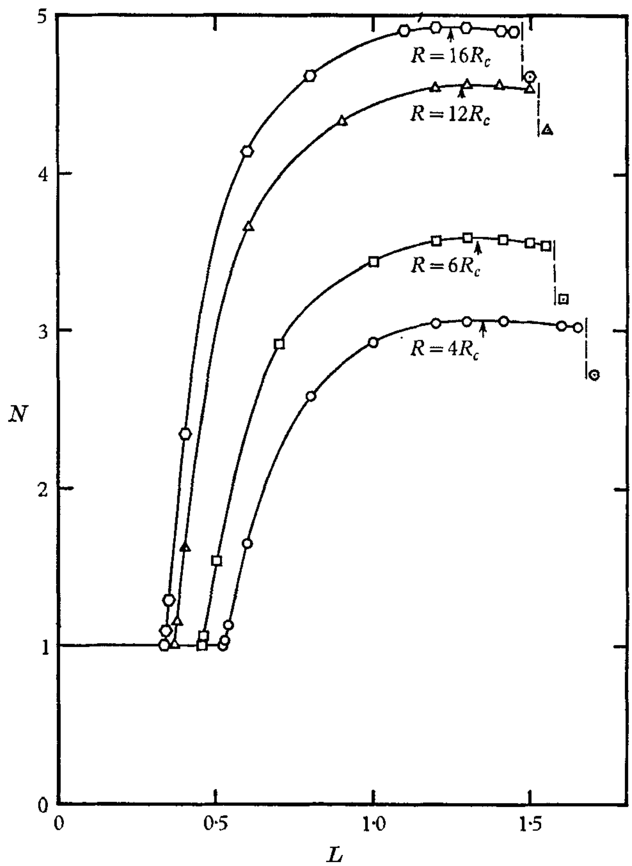

FIGURE 2. Steady-state Nusselt number *versus* length ratio for various Rayleigh numbers.

A plot of Nusselt number reached in the steady state *versus* length ratio for various Rayleigh numbers is seen in figure 2. The approximate location of the maximum Nusselt number for each Rayleigh number is indicated by an arrow. It is seen that the length ratio for maximum heat transfer decreases slightly with increasing Rayleigh number. The discontinuities indicated by the dashed lines on the right side of the figure represent the change from the development of a one-cell (where 'cell' is taken to mean a single roll) to a two-cell configuration when infinitesimal initial disturbances are used. It is seen that the transition length ratio decreases slightly with increasing Rayleigh number.

Figure 3 shows additional calculations for a Rayleigh number of 2630 ($4R_c$). The circles and solid line indicate the steady-state Nusselt number obtained when infinitesimal initial disturbances are used. It is seen that for increased length ratio the Nusselt number decreases sharply at the points where the number of cells increases. The magnitude of this decrease decreases for changes between increasingly larger numbers of cells. In addition, it appears that the maximum Nusselt number reached is nearly the same no matter how many cells form.

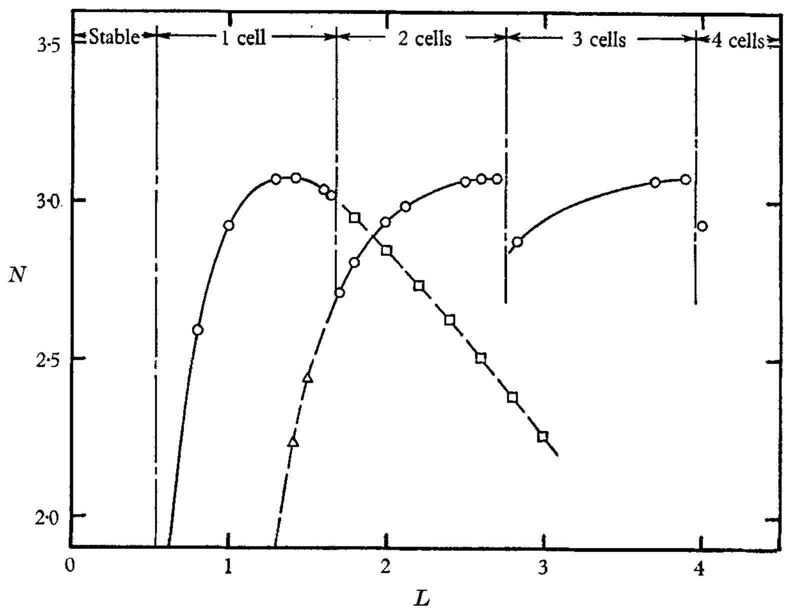

FIGURE 3. Steady-state Nusselt number *versus* length ratio for $R = 4R_c$. The length ratio regions for solutions with various numbers of cells are shown for initially infinitesimal disturbances.

The squares in figure 3 indicate the results of calculations in which the Fourier coefficients of the velocity disturbance and of the temperature which resulted from the steady-state solution for the next smaller length ratio starting with $L = 1{\cdot}65$ (a one-cell solution) were used as initial conditions. One might look on this procedure as a sort of adiabatic stretching apart of the lateral boundaries. The results were somewhat surprising in that the system would not go over to a two-cell solution even when the heat transfer for the one-cell solution was markedly less than that obtainable for the two-cell solution at that length ratio. Only when the length ratio was increased to greater than 3·0 did the system start to transform, and then into a three-cell solution. The transformation took place so very slowly that it was not possible to accurately fix the length ratio where it first occurred.

The triangles in figure 3 indicate the results of calculations in which the Fourier coefficients that resulted from the steady-state solution for the next larger length ratio starting with $L = 1{\cdot}7$ (a two-cell solution) were used as initial conditions, a sort of adiabatic pushing together of the lateral boundaries. In this case the system would not go over to a one-cell solution until the length ratio was decreased to smaller than about 1·06, which is the limit for the formation of two equal cells according to linear theory. At $L = 1\cdot 0$, for example, the two-cell solution decayed rapidly until the velocity disturbance was very small and the Nusselt number very nearly unity; then the one-cell solution grew until it obtained its normal velocity distribution and Nusselt number.

On examining the Fourier components of the velocity, $A_{mn}(t)$, for the one-cell solution, one finds that, as time increases indefinitely, the components for which $m+n$ is odd become vanishingly small. In the two-cell solution the components for which $m+n$ is even become vanishingly small as time increases indefinitely. If the $m+n$ odd components are set exactly equal to zero, only an odd number of cells can form, and these components always remain zero. Similarly, if the $m+n$ even components are set exactly equal to zero, only an even number of cells can form. The mathematical reason for this behaviour can be seen if one examines (22). Here one sees that growth of one class ($m+n$ odd or even) of Fourier components will not occur unless at least one component of that same class is nonzero. In general, it appears that transitions between even- and odd-cell solutions are inhibited when one class of solutions has sufficiently larger-amplitude Fourier components than the other class.

An investigation was made to determine the ratio of amplitudes of the Fourier components of odd- and even-cell solutions that is necessary to cause one or the other class of solution to develop from small disturbances. One sees in figure 3 for $R = 4R_c$ that at a length ratio of about $L = 1\cdot 9$ the Nusselt numbers for one- and two-cell solutions are equal. Thus any possible tendency for the system to move to a state of maximum heat transfer will have a minimum effect at this length ratio. Keeping both A_{11} and A_{12} small (10^{-2} or less) and setting the rest of the Fourier components of the velocity, A_{mn}, equal to 10^{-10}, the ratio of the initial values of A_{11} to A_{12} was varied. It was found that, if the initial ratio of A_{11} to A_{12} was about 10^2 or greater, the one-cell solution resulted. The apparent reason for this behaviour is that, until the components reach an amplitude of about 1 (for $R = 4R_c$), the non-linear interactions are negligible and the various horizontal wave-numbers present in the horizontal Fourier expansion develop uncoupled. The horizontal wave-number that first attains sufficient amplitude to cause non-linear interactions such that the temperature field is modified will effectively be 'locked in' and will dominate the flow pattern unless it is very inappropriate for the length ratio involved—for example, an initially dominant one-cell solution at a length ratio of $3\cdot 5$ does not persist but transforms into a three-cell solution. At a length ratio of $1\cdot 9$ the principal Fourier component for a two-cell solution is amplified at a much faster rate than that for a one-cell solution; thus, unless the initial amplitude of A_{11} is about 10^2 times greater, A_{12} will attain the dominating amplitude first and a two-cell solution will result.

6. Results for a sudden change in temperature

In order to further explore the horizontal wavelength selection process for this system some calculations were made with an initial temperature profile which resulted from starting with an isothermal fluid and then suddenly changing the temperature at one of the horizontal boundaries (increasing the bottom or decreasing the top temperature) by an increment ΔT at $t = 0$. This temperature profile is generated by setting the $C_l(0) = 1$ in (21). The system was investigated in a manner identical with that used for the system with an initially linear profile. The behaviour of the system, however, was somewhat different, as can be seen in figure 4. The Nusselt number for the surface at which the temperature was suddenly changed decreased rapidly just after $t = 0$ and then slowly decreased

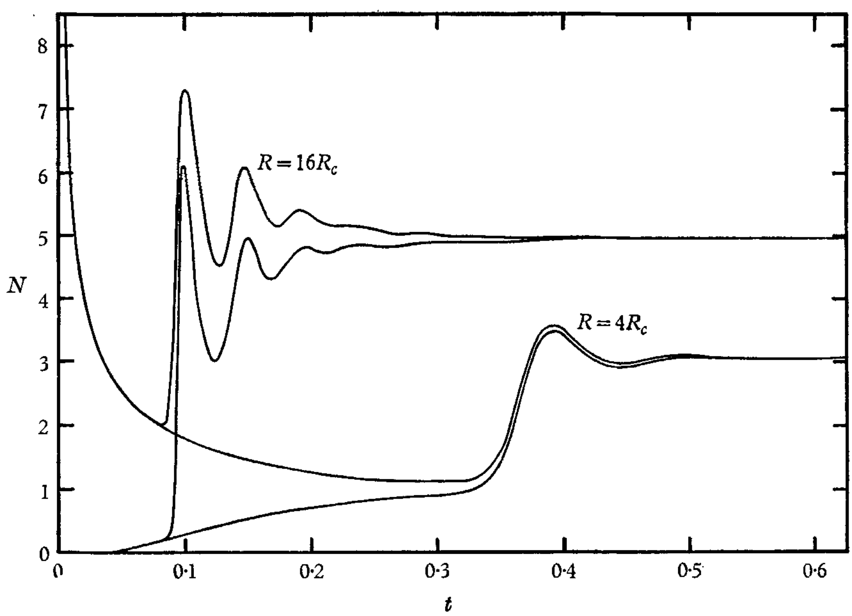

FIGURE 4. Time development of Nusselt number starting from a sudden change in temperature and an infinitesimal disturbance.

towards unity until onset of convection took place; the Nusselt number for the unchanged surface increased very slowly towards unity until onset of convection. At low Rayleigh numbers, $R = 4R_c$, the Nusselt numbers both became almost unity and the temperature profile almost linear before onset of convection. Thus at low Rayleigh numbers the effect of the initial temperature profile upon the subsequent convection was negligible. For larger Rayleigh numbers, $R = 16R_c$, the effects of the initial temperature profile become more important since onset of convection occurs before the profile has time to become linear.

A series of calculations for various length ratios at $R = 16R_c$ was made to see if the length ratio for the transition from development of one-cell to the development of two-cell solutions was affected by the initial temperature profile. It was found that the transition occurred between length ratios of 1·40 and 1·45 for the case of a sudden change in temperature compared with between 1·45 and 1·50 for the case of an initially linear temperature profile. According to the linear theory for the case of a sudden change in temperature (Foster 1965*a*) the wavelengths of the disturbances which are amplified the most at onset of convection are independent of the depth of the fluid layer and are inversely proportional to the cube root of ΔT when the Rayleigh number exceeds about 10^4. Thus for larger Rayleigh numbers smaller wavelength disturbances are amplified more at onset of convection, and the decrease of the transition length ratio for larger Rayleigh numbers may be explained by the 'locking in' phenomenon discussed in §5.

7. The case of infinite horizontal extent

In order to apply Fourier analysis to the investigation of a layer of fluid which has no lateral boundaries one must assume that the variables are exactly periodic; however, the interval over which this periodicity takes place is not known *a priori*. One can only argue that experiments show that periodic convection cells may form, and for convection apparatus with very large horizontal length to depth ratios the periodicity is independent of the length ratio. The choice of the interval over which periodicity occurs will be discussed later.

The Fourier series must now include both sine and cosine terms since there are no lateral boundary surfaces which place restrictions on the allowed functions. Thus the expansion for the vertical velocity disturbance becomes

$$w = \sum_m \sum_n [A_{mn}(t) \cos(max) + B_{mn}(t) \sin(max)] \sin(n\pi z), \tag{25}$$

where a is the dimensionless wave-number of the periodicity. Solving (3) for u we now have

$$u = -\sum_m \sum_n (n\pi/ma) [A_{mn}(t) \sin(max) - B_{mn}(t) \cos(max)] \cos(n\pi z), \tag{26}$$

and solving (11) for θ gives

$$\theta = -\sum_m \sum_n \frac{(n^2\pi^2 + m^2a^2)^2}{Rm^2a^2} [A_{mn}(t) \cos(max) + B_{mn}(t) \sin(max)] \sin(n\pi z). \tag{27}$$

The expansions for w, u, θ and $\overline{T}$ are substituted into the heat conduction equation (6) and sets of coupled ordinary differential equations for the Fourier coefficients $A_{mn}(t)$, $B_{mn}(t)$ and $C_l(t)$ are obtained in a manner similar to that used in §4. In the present case these sets of equations are much more complicated than (22) and (23) since there are cross-terms involving products of $A_{mn}(t)$ and $B_{mn}(t)$ as well as $N \times M$ additional equations for $B_{mn}(t)$. The memory of the computer limited calculations for this vastly complicated system to those for $N = M = 6$ or less terms. This in turn limited accurate calculations to Rayleigh numbers less than about $R = 6R_c$.

A series of calculations for $R = 4R_c$ was carried out starting with a linear temperature profile. The procedure followed was identical to that used in §5 for the case of finite horizontal extent. If we consider the dimensionless 'periodicity half-length' to be equal to π/a, then a plot of Nusselt number *versus* periodicity half-length for the infinite case can be compared with the plot of Nusselt number *versus* length ratio for the finite case. It was found that the two plots were very similar. The periodicity half-length in the infinite case and the length ratio in the finite case for maximum Nusselt number for one-cell solutions were very nearly the same, about 1·35, though the maximum Nusselt number for the infinite case was slightly higher, about 3·21, than that for the finite case, about 3·07.

Physically, the effect of the lateral boundaries is to place a constraint on the allowable motions of the fluid, and, in general, such a constraint would be expected to cause a reduction in the heat transfer of the system. Apparently the only restriction that 'free', insulated lateral boundaries can impose upon the system is that the flow at the lateral boundaries is required to be vertical. In the case of infinite horizontal extent it was found that the lateral edges of the cells were not vertical but were slightly inclined so that alternate cells were somewhat wider at the top than at the bottom with the cells in between wider at the bottom than at the top. In the finite case the cells seemed to be symmetrical as far as could be determined in the cases of two and three cells. Since the lateral boundaries exert a strong influence on the flow pattern when the number of cells is small, it is not surprising that the lateral edges of the cells did not tilt. Unfortunately we could not carry out accurate calculations for configurations of more than four cells in the finite case, but we can speculate that, if the distance between the lateral boundaries were greatly increased so that a very large number of cells would form, then the influence of the lateral boundaries on the interior cells would be small. In this case the lateral edges of the interior cells might become tilted and the Nusselt number might increase and approach the value calculated for no lateral boundaries.

Mathematically, the effect of the lateral boundaries is to permit only cosine terms in the Fourier expansion for the vertical velocity. If in the Fourier expansion for w, equation (25), one retains only the sine terms, then the only non-linear terms occur in the sets of equations for $B_{mn}(t)$ and $C_l(t)$ similar to (23). These are the non-linear terms which represent the interaction of the fluctuations with the mean temperature field. The non-linear terms which represent the fluctuating self-interactions in (22) do not appear when the expansion for w (for infinite Prandtl number) includes only sine terms since the sine functions are orthogonal in the integrals involved. Calculations for the sine expansion of w show a Nusselt number *versus* periodicity half-length relation very similar to the two other expansions. The maximum Nusselt number for $R = 4R_c$ again occurred at about 1·35 with a magnitude of about 3·19. This is close to, but slightly less than, that for the complete sine–cosine expansion, which is to be expected since the incomplete expansion imposes a constraint on the possible fluid motions. It is interesting to note that in the complete expansion solution the principal sine function is favoured; at maximum Nusselt number for $R = 4R_c$ and a one-cell solution, the ratio B_{11}/A_{11} is 1·8. The mathematical reason for the favouring of the sine functions can be seen by examining the non-linear terms from the fluctuating self-interactions in the analogue to (22). Only the cross-terms $A_{mn}(t)B_{ij}(t)$ appear in the equations for the sine coefficients while both the $A_{mn}(t)A_{ij}(t)$ and $B_{mn}(t)B_{ij}(t)$ terms appear in the equations for the cosine coefficients. Since the net effect of the fluctuating self-interactions is to inhibit the fluctuations, the cosine coefficients are suppressed.

The periodicity half-length at which solutions for one cell changed to solutions for two cells occurred at approximately the same point as in the case of finite length. Ideally, the analysis should be carried out at successively longer periodicity lengths so that the preferred cell size could be ascertained with greater accuracy by noting the transition periodicity length for a large number of cells; however, the restricted memory of the computer and the limited computer time available prevented extending the analysis beyond the first transition.

8. Discussion of results

The present investigation shows that at small Rayleigh numbers and infinite Prandtl number the number and size of convection cells that form for a specified Rayleigh number depend on the ratio of the distance between the lateral boundaries to the depth of the fluid layer and on the initial conditions. For example, at $R = 4R_c$ and $L = 1{\cdot}8$ an initially infinitesimal 'white noise' disturbance will grow into a two-cell solution and will in the steady state have a Nusselt number of about 2·80 even though a one-cell solution for the same conditions would have a steady-state Nusselt number of 2·95. Thus the hypothesis that the system should tend to one in which heat transport is a maximum is not valid at low Rayleigh numbers for two-dimensional flow. There are under certain conditions more than one metastable solution which can develop depending upon the initial conditions.

Busse (1967) has shown in the case where there are no lateral boundaries that below a Rayleigh number of about 20,000 two-dimensional convection should be stable over a range of wavelengths. Chen & Whitehead (1968) have shown experimentally that it is possible to artificially induce two-dimensional convection cells which are stable over a range of wavelengths. A more natural way to induce a range of wavelengths, which was discussed in § 6, is to rapidly change the temperature at one horizontal surface. In this case it is possible for the fluid in a relatively thin boundary layer to become unstable for a wavelength related to the boundary-layer thickness and independent of the depth of the fluid layer. Experiments (Foster 1965*b*) have shown that the wavelengths of disturbances that are amplified the most at onset of convection determine the wavelengths of the initial convection cells. Provided these wavelengths are within the stable band for the Rayleigh number involved, these wavelength convection cells should persist. As long as they are small, the initial disturbances proceed to grow uncoupled, and the phenomenon is essentially linear. Thus the linear theory can be extremely useful in the prediction of convection cell size since it can provide realistic initial conditions for the non-linear theory. This is particularly true in the case where there are no lateral boundaries because in this case there is no *a priori* horizontal length scale.

The basic reason for the failure of the principle of maximum heat transport is that convection at low Rayleigh number is an organized steady flow which is dependent upon its past history. At large Rayleigh numbers the flow becomes turbulent and may lose any memory of its past history. In this case, perhaps statistical mechanical considerations can be used to determine the characteristic eddy size in the steady state. However, in geophysical fluids, such as the ocean and the atmosphere, it is likely that even turbulent convection may exhibit eddy size distributions that depend upon the initial conditions since it is quite common to find systems which are not in a steady state in nature.

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